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The translational and rotational motions of an n-dimensional hypersphere through a viscous fluid at small Reynolds numbers

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Expressions are derived for the n-dimensional Stokes velocity and pressure fields (and stream function) corresponding, respectively, to the translation and rotation of a hypersphere in a viscous fluid at rest at infinity. These are utilized to calculate the force and first antisymmetric stress moment ('torque') on the n-dimensional hypersphere. They are also utilized to derive the generalizations of Faxén's laws for the force and stress moment corresponding to arbitrarily prescribed velocity fields on the hypersphere surface and at infinity.

1. Introduction

The spectacular success of applied mathematical developments during the past decade concerned with critical-point phenomena (Wilson 1971, 1974; Wilson & Fisher 1972) and other complex physical processes (Wilson & Kogut 1974; Pfeuty & Toulouse 1977; Amit 1978; de Gennes 1979; Witten 1980) has pointed out the advantages of regarding the number n of spatial dimensions characterizing a physical problem as a variable parameter rather than as the fixed numbers 2 or 3. Introduction of this artificial parameter permits the development of perturbation schemes that often yield fresh mathematical insights into classical physical problems and their underlying scaling laws.

Low-Reynolds-number hydrodynamics would appear to provide an exceptionally fertile field for application of these new ideas, since questions of dimensionality and scaling play central roles in the existence of several well-known paradoxes connected with solutions of the Navier–Stokes equations in the limit $R \to 0$. Thus, Stokes's paradox (Krakowski & Charnes 1951; Finn & Noll 1957; Kaplun 1957; Proudman & Pearson 1957; Chang 1961) in two dimensions and Whitehead's paradox (Kaplun & Lagerstrom 1957; Proudman & Pearson 1957) in three dimensions arise from the existence of a singularity at infinity in the perturbation flow field of an otherwise uniform stream caused by the presence of an obstacle. Here, the requisite scaling in the distant flow field is essentially one of stretching the dimensionless distance r/a from the centre of the body (a is the body radius and r is the radial distance) via multiplication by the Reynolds number $R = aU/\nu$ (U = free-stream velocity, ν = kinematic viscosity). The weakening nature of this singularity in passing from $n = 2$ to $n = 3$ dimensions is such that whereas no solution of the Navier–Stokes equations (satisfying the boundary conditions at infinity) exists at $R = 0$ in two dimensions, a solution does exist in three dimensions. In the latter case the singularity at infinity is not revealed until the inclusion of terms of the first order in R (Oseen 1927). Thus, it is natural to inquire into the apparently monotonic weakening of the nature of the singularity with increasing n. In particular, does there exist any finite value of n beyond which the singularity disappears entirely?

In the related context of rotational, rather than translational, problems it is known (Pai 1956; Landau & Lifshitz 1959) that the two-dimensional $R = 0$ solution for a rotating circular cylinder ($R = a^2\omega/\nu$, where ω is the angular velocity) represents an exact solution of the full Navier–Stokes equations for any R (provided that the $R = 0$ pressure field is supplemented by the addition of a centrifugal pressure field), whereas the corresponding three-dimensional solution for a rotating sphere (Landau & Lifshitz 1959) is not an exact Navier–Stokes solution. However, the latter $n = 3$ solution does represent a uniformly valid solution of the Navier–Stokes equations at small Reynolds numbers (see the detailed review of Brenner 1966*b*, pp. 358–359), in contrast with the comparable $n = 3$ translational solution – where Oseen's, rather than Stokes's, equations provide a uniformly valid solution as $R \to 0$.

These observations suggest the potential utility of solutions of the basic translational and rotational solutions of the Stokes equations for a hypersphere moving within an n-dimensional viscous fluid which is at rest at infinity. These fundamental solutions are derived in subsequent sections. As an easily claimed bonus we also give the generalizations of Faxén's laws for the hypersphere.

Readers not specifically interested in the esoterics of n-dimensional Stokes flows will, nevertheless, find here the outline of a novel geometrical scheme for solving conventional two- and three-dimensional Stokes flow and related *linear* problems.† This general technique takes maximum advantage of linearity and geometric-symmetry arguments to reduce the overall problems from the realm of vector partial-differential equations to scalar total differential equations.

2. Geometry of *n*-dimensional space

While it may appear convenient, or even necessary, to use n-dimensional hyperspherical polar co-ordinates $(r, \theta, \phi_1, \phi_2, \ldots, \phi_{n-2})$ (Bateman 1944; Sommerville 1958; Sommerfeld 1964) in the formulation and solution of the hypersphere boundary-value problems posed, we nevertheless shall not require this. Rather, it will suffice to consider only the more elementary Cartesian system $(x_1, x_2, \ldots, x_n)$ (Bateman 1944; Sommerville 1958), with corresponding orthonormal unit vectors $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \ldots, \hat{\mathbf{x}}_n)$. These satisfy $\hat{\mathbf{x}}_k \cdot \hat{\mathbf{x}}_l = \delta_{kl}$, with δ_{kl} the Kronecker delta.

In this system the position vector $\mathbf{r}$ possesses the representation

$$\mathbf{r} = \hat{\mathbf{x}}_1 x_1 + \hat{\mathbf{x}}_2 x_2 + \ldots + \hat{\mathbf{x}}_n x_n \equiv \hat{\mathbf{x}}_j x_j \tag{1}$$

(summation convention on repeated indices). The corresponding n-dimensional gradient operator and Laplacian are, respectively,

$$\nabla = \hat{\mathbf{x}}_1 \partial/\partial x_1 + \hat{\mathbf{x}}_2 \partial/\partial x_2 + \ldots + \hat{\mathbf{x}}_n \partial/\partial x_n \equiv \hat{\mathbf{x}}_j \partial/\partial x_j, \tag{2}$$

and

$$\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \ldots + \partial^2/\partial x_n^2 \equiv \partial^2/\partial x_j \partial x_j. \tag{3}$$

† The scheme superficially resembles that of Landau & Lifshitz (1959, pp. 63 and 68). However, our technique does not use any non-geometrical 'tricks' to eliminate the subsidiary condition that the vector field be divergence-free.

In this representation the dyadic idemfactor $\mathbf{I} = \nabla \mathbf{r}$ is

$$\mathbf{I} = \hat{\mathbf{x}}_1 \hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2 \hat{\mathbf{x}}_2 + \ldots + \hat{\mathbf{x}}_n \hat{\mathbf{x}}_n \equiv \hat{\mathbf{x}}_k \hat{\mathbf{x}}_l \delta_{kl}. \tag{4}$$

With

$$r = |\mathbf{r}| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}, \tag{5}$$

the equation of the surface of a hypersphere of 'radius' a is $r = a$. In this notation

$$\hat{\mathbf{r}} = \mathbf{r}/r \equiv \hat{\mathbf{x}}_j x_j / r \tag{6}$$

denotes a unit radial vector.

The scalar element of surface 'area' ds on an n-dimensional hypersphere of radius r is given by (Bateman 1944; Sommerfeld 1964)

$$ds = r^{n-1} d\lambda, \tag{7}$$

where $d\lambda$ is an areal element on an n-dimensional hypersphere of unit radius.† The surface area

$$S = \int_{\Lambda_1} ds \Big|_{r=a}$$

of a hypersphere of radius a is given explicitly as

$$S = a^{n-1} \Lambda, \tag{8}$$

where (Bateman 1944; Sommerville 1958; Sommerfeld 1964)

$$\Lambda = \int_{\Lambda_1} d\lambda = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}, \tag{9}$$

in which Λ_1 denotes integration over the unit hypersphere. For integer dimensions the gamma function (Abramowitz & Stegun 1964) possesses the properties that

$$\Gamma(m+1) = m!,$$

$$\Gamma(m+\tfrac{1}{2}) = \frac{1 \,.\, 3 \,.\, 5 \ldots (2m-1)}{2^m} \pi^{\frac{1}{2}}.$$

The directed element of surface area on the hypersphere is given by

$$d\mathbf{s} = \hat{\mathbf{r}}\, ds \tag{10}$$

or, equivalently, in Cartesian tensor notation,

$$d\mathbf{s} = \hat{\mathbf{x}}_j\, ds_j \quad \text{and} \quad ds_j = (x_j / r)\, ds. \tag{11}$$

† We shall not require an explicit representation (Bateman 1944; Sommerfeld 1964) of ds in the hyperspherical polar co-ordinate system $(r, \theta, \phi_1, \phi_2, \ldots, \phi_{n-2})$, though it can in fact be calculated from knowledge of the metrical coefficients (Happel & Brenner 1973) of this orthogonal curvilinear co-ordinate system. While, as shown by Sommerfeld (1964), the value of Λ (and hence S) given in (9) can be obtained by integration of this hyperspherical polar representation of ds, the result can also be obtained (Sommerville 1958, p. 135) by purely geometrical recursive arguments which do not require such a representation. We stress this point to emphasize the fact that our calculational scheme is purely geometric and does not explicitly or implicitly rely upon the use of any co-ordinate systems, be they hyperspherical polar or Cartesian. Though it may appear that we are, in fact, explicitly utilizing the Cartesian system in subsequent calculations, this is not really true, as the interested reader may confirm by reverting from Cartesian tensor notation (x_j) to invariant vector notation $(\mathbf{r})$.

3. Translation of a hypersphere

With $\mathbf{v}(x_1, x_2, \ldots, x_n)$ and $p(x_1, x_2, \ldots, x_n)$ the velocity vector and pressure fields at a point $\mathbf{r}(x_1, x_2, \ldots, x_n)$ of the hyperspace, the creeping motion and continuity equations possess their usual forms

$$\nabla^2 v_j = (1/\mu)\, \partial p/\partial x_j \tag{12}$$

and

$$\partial v_j/\partial x_j = 0, \tag{13}$$

where μ is the viscosity.

In the case of a hypersphere of radius a translating with constant velocity $\mathbf{U}$ through a fluid at rest at infinity, it is required that

$$v_k = U_k \quad \text{at} \quad r = a, \tag{14}$$

$$v_k \to 0 \quad \text{as} \quad r \to \infty, \tag{15}$$

and

$$p \to p_\infty \quad \text{as} \quad r \to \infty, \tag{16}$$

where p_∞ is the uniform pressure at infinity. The first of these equations presupposes the usual adherence condition at a fluid–solid interface.

In consequence of the linearity of the differential equations and boundary conditions, we may define (Brenner 1964*a*) a dyadic 'velocity' field

$$V_{jk}(x_1, x_2, \ldots, x_n) \equiv \mathbf{V}(\mathbf{r})$$

and a vector 'pressure' field $P_k(x_1, x_2, \ldots, x_n) \equiv \mathbf{P}(\mathbf{r})$ via the linear relations

$$v_j = V_{jk} U_k \quad (\mathbf{v} = \mathbf{V}.\mathbf{U}), \tag{17}$$

$$p - p_\infty = \mu P_k U_k \quad (p - p_\infty = \mu \mathbf{P}.\mathbf{U}). \tag{18}$$

Upon substituting these into (12)–(16) and eliminating the constant vector $\mathbf{U}$ from the resulting expressions, the tensor velocity and pressure fields (V_{jk}, P_k) are found to satisfy the following differential equations:

$$\nabla^2 V_{jk} = \partial P_k/\partial x_j, \tag{19}$$

$$\partial V_{jk}/\partial x_j = 0, \tag{20}$$

and boundary conditions:

$$V_{jk} = \delta_{jk} \quad \text{at} \quad r = a, \tag{21}$$

$$V_{jk} \to 0 \quad \text{as} \quad r \to \infty, \tag{22}$$

$$P_k \to 0 \quad \text{as} \quad r \to \infty. \tag{23}$$

Equations (19) to (23) are purely *geometric* relations. Specifically they show that the fields V_{jk} and P_k are wholly geometric in nature, being independent of the magnitude and direction of the parameter $\mathbf{U}$ and of the scalar μ. As such, at any field point $\mathbf{r}$ these higher-order tensor fields depend only upon the geometry of the hyperspherical particle. (The n-dimensional Euclidean space is, of course, a flat space, so that there exist no hidden geometrical parameters characterizing the metrical properties of this space, such as its curvature.) An immediate consequence of this fact is that the geometric symmetries of the tensor fields V_{jk} and P_k must be the same as that of the hypersphere itself. As such, the fields $(\mathbf{V}, \mathbf{P})$ must be expressible solely in terms of the position vector $\mathbf{r}$ (and its tensor invariants, such as $r = |\mathbf{r}|$ and the dyadic idemfactor $\mathbf{I} = \nabla \mathbf{r}$). This observation provides the *ansatz* necessary to solve the problem.

The only *vector* field which can be derived from $\mathbf{r}$ is $\mathbf{r}$ itself and scalar multiples thereof, which can at most be functions of r. Thus, based upon the prior geometric arguments, $\mathbf{P}$ must necessarily be of the form

$$\mathbf{P} = \mathbf{r}f(r), \quad \text{i.e.} \quad P_k = x_k f(r), \tag{24}$$

where f is a scalar function of r to be determined.

Similarly, the only *dyadic* fields which can be formed from $\mathbf{r}$ and its invariants are $\mathbf{rr}$ and $\mathbf{I}$, and scalar multiples thereof, which can depend at most upon r. This requires *a priori* that $\mathbf{V}$ be of the form

$$\mathbf{V} = \mathbf{I}g(r) + \mathbf{rr}h(r), \quad \text{i.e.} \quad V_{jk} = \delta_{jk} g(r) + x_j x_k h(r), \tag{25}$$

with g and h scalar functions of r to be determined.

The differential equations and boundary equations satisfied by the scalar fields f, g and h are readily determined with the aid of the following elementary Cartesian tensor identities:

$$\partial x_j / \partial x_k = \delta_{jk}, \quad \delta_{kk} = n,$$

and

$$\partial F(r)/\partial x_j = (x_j/r)\, dF(r)/dr.$$

In the first place, (19) and (20) require that

$$\nabla^2 P_k = 0. \tag{26}$$

But, from (24),

$$\nabla^2 P_k = \nabla^2(x_k f) = x_k \left[\nabla_r^2 f + \frac{2}{r}\frac{df}{dr}\right], \tag{27}$$

where (Bateman 1944; Sommerfeld 1964)

$$\nabla_r^2 = \frac{1}{r^{n-1}} \frac{d}{dr}\left(r^{n-1}\frac{d}{dr}\right) \tag{28}$$

is the hyperspherically symmetric n-dimensional Laplace operator. Equation (26) requires that the term in square brackets appearing on the right-hand side of (27) be zero identically. Integration easily yields the general solution

$$f = C_1 r^{-n} + C,$$

with C_1 and C integration constants to be determined. Condition (23) requires that $C = 0$, whence we obtain

$$P_k = C_1 x_k r^{-n}. \tag{29}$$

Thus, except for the numerical value of C_1, the pressure field is uniquely determined independently of the velocity field as a consequence of the required geometric symmetry.

From equations (25) and (28) there follows

$$\begin{aligned}\nabla^2 V_{jk} - \partial P_k/\partial x_j = &\left[\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{dg}{dr}\right) + 2h - C_1 r^{-n}\right]\delta_{jk} \\ &+ \left[\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{dh}{dr}\right) + \frac{4}{r}\frac{dh}{dr} + nC_1 r^{-(n+2)}\right] x_j x_k.\end{aligned} \tag{30}$$

Hence, (19) requires that each of the two above terms in square brackets vanish separately. The second of the resulting equations involves only the function $h(r)$, which therefore may be solved independently of the first to obtain

$$h = \tfrac{1}{2}C_1 r^{-n} + C_2 r^{-(n+2)} + C_3, \tag{31}$$

with C_2 and C_3 new integration constants to be determined. This result may be substituted into the first of the pair of equations resulting from the vanishing of (30) to obtain the total differential equation satisfied by $g(r)$. The resulting equation for g is readily integrated to obtain

$$g = -n^{-1}C_2 r^{-n} - n^{-1}C_3 r^2 - (n-2)^{-1}C_4 r^{-(n-2)} + C_5, \tag{32}$$

provided that $n \neq 2$. Here, C_4 and C_5 are two new additional integration constants to be determined.

Except for the algebraic determination of the five constants C_1 to C_5, the problem is now solved. It remains yet to satisfy the continuity equation (20). From (25) it easily follows that

$$\frac{\partial V_{jk}}{\partial x_j} = \left[\frac{1}{r}\frac{dg}{dr} + r\frac{dh}{dr} + (n+1)h\right] x_k.$$

Introduction of (31) and (32) into this relation thereby yields

$$\frac{\partial V_{jk}}{\partial x_j} = \left[\left\{(n+1) - \frac{2}{n}\right\} C_3 + r^{-n}\left(C_4 + \frac{C_1}{2}\right)\right] x_k.$$

Hence, the continuity equation (20) will be satisfied provided that

$$C_3 = 0, \quad 2C_4 + C_1 = 0. \tag{33}$$

Upon collecting together those results pertaining to the tensor velocity field we have thus far that

$$V_{jk} = \left[\frac{C_1}{2(n-2)} r^{-(n-2)} - \frac{1}{n}C_2 r^{-n} + C_5\right]\delta_{jk} + \left[\frac{C_1}{2} r^{-n} + C_2 r^{-(n+2)}\right] x_j x_k \quad (n > 2). \tag{34}$$

Satisfaction of the boundary condition (22) at infinity requires that $C_5 = 0$. The remaining boundary condition (21) on the hypersphere surface requires that, at $r = a$, the first term in square brackets in (34) have the value unity and that the second term in square brackets be zero. In this manner the two constants C_1 and C_2 are found to have the values

$$C_1 = \frac{n(n-2)}{(n-1)} a^{n-2}, \quad C_2 = -\frac{n(n-2)}{2(n-1)} a^n. \tag{35}$$

In summary, the solutions of the tensor Stokes equations for a translating hypersphere are, in invariant form,

$$\mathbf{V} = \frac{1}{2(n-1)}\left(\frac{a}{r}\right)^n \left[n\left(\frac{r}{a}\right)^2 + n - 2\right]\mathbf{I} + \frac{n(n-2)}{2(n-1)}\left(\frac{a}{r}\right)^n\left[\left(\frac{r}{a}\right)^2 - 1\right]\hat{\mathbf{r}}\hat{\mathbf{r}}, \tag{36}$$

and

$$\mathbf{P} = \frac{n(n-2)}{(n-1)}\frac{1}{a}\left(\frac{a}{r}\right)^{n-1}\hat{\mathbf{r}}, \tag{37}$$

for $n > 2$, with $\hat{\mathbf{r}}$ defined as in (6). From (17) and (18) the physical velocity and pressure fields are given by

$$\mathbf{v} = \frac{U}{2(n-1)}\left(\frac{a}{r}\right)^n\left[n\left(\frac{r}{a}\right)^2+n-2\right]\hat{\mathbf{U}}+\frac{n(n-2)}{2(n-1)}U\left(\frac{a}{r}\right)^n\left[\left(\frac{r}{a}\right)^2-1\right]\hat{\mathbf{r}}\hat{\mathbf{r}}.\hat{\mathbf{U}} \tag{38}$$

and
$$p = p_\infty+\frac{n(n-2)}{n-1}\frac{\mu U}{a}\left(\frac{a}{r}\right)^{n-1}\hat{\mathbf{r}}.\hat{\mathbf{U}}, \tag{39}$$

where
$$U = |\mathbf{U}| \tag{40}$$

is the magnitude of the translational velocity vector, and

$$\hat{\mathbf{U}} = \mathbf{U}/U \tag{41}$$

denotes a unit vector in the direction in which the hypersphere translates. Alternatively, in Cartesian tensor form,

$$v_j = \frac{1}{2(n-1)}\left(\frac{a}{r}\right)^n\left[n\left(\frac{r}{a}\right)^2+n-2\right]U_j+\frac{n(n-2)}{2(n-1)}\left(\frac{a}{r}\right)^n\left[\left(\frac{r}{a}\right)^2-1\right]\frac{x_j x_k U_k}{r^2}, \tag{42}$$

and
$$p = p_\infty+\frac{n(n-2)}{(n-1)}\frac{\mu}{a}\left(\frac{a}{r}\right)^{n-1}\frac{x_k U_k}{r}. \tag{43}$$

In the case $n = 3$ these correctly give the usual Stokes velocity and pressure fields for a translating sphere (Happel & Brenner 1973). For $n = 2$ Stokes paradox is clearly in evidence, since in that case equation (38) fails to fulfil the requirement that $\mathbf{v}$ vanish at $r = \infty$. It is interesting to observe that the paradox does not exist in a space of $n = 2+\epsilon$ dimensions, where ϵ may be taken to be arbitrarily small.

Equations (38) and (39) show that $\mathbf{v}/U \sim O(a/r)^{n-2}$ and $(p-p_\infty)\,a/\mu U \sim O(a/r)^{n-1}$ as $r/a \to \infty$. Thus, the greater the dimensionality of the space the more rapidly is the disturbance generated by the moving hypersphere attenuated.

Present results for the velocity field are recast in the form of an n-dimensional stream function in § 6.

3.1. *Force on the hypersphere*

We proceed to derive the analogue of Stokes's law for the force exerted by the fluid on the translating hypersphere. This vector force is given by

$$F_j = \int_{\Lambda_1} ds_i\,\pi_{ij}\Big|_{r=a}, \tag{44}$$

where
$$\pi_{ij} = -\delta_{ij}p+\mu(\partial v_j/\partial x_i+\partial v_i/\partial x_j) \tag{45}$$

is the Newtonian stress tensor for an incompressible fluid, and ds_i is given in (11). From (42) and (43) we readily obtain

$$ds_i\,\pi_{ij}|_{r=a} = -\frac{n(n-2)}{n-1}\mu a^{n-2}U_j\,d\lambda \tag{46}$$

upon suppressing the irrelevant constant p_∞. With use of (9), we find the vector force obtained upon integration of (44) to be

$$\mathbf{F} = -\frac{2n(n-2)\,\pi^{\frac{1}{2}n}}{(n-1)\,\Gamma(\frac{1}{2}n)}\mu a^{n-2}\mathbf{U}. \tag{47}$$

In agreement with expectations, this force acts in a direction opposite to that in which the hypersphere translates.

For $n = 3$ this properly reduces to Stokes's law, $\mathbf{F} = -6\pi\mu a\mathbf{U}$. It is clearly degenerate for $n = 2$, corresponding to Stokes paradox.

It is perhaps more meaningful to express (47) in the form of the force per unit of surface 'area' S [cf. equations (8) and (9)], namely

$$\frac{\mathbf{F}}{S} = -\frac{n(n-2)\mu\mathbf{U}}{(n-1)a}. \tag{48}$$

This result shows that, all other things being equal, the force per unit area diminishes with increasing radius a and increases with increasing n. The latter obviously arises from the fact that the velocity gradients causing the stress increase with increasing n owing to the more rapid decay of velocity with distance – from the value $\mathbf{v} = \mathbf{U}$ on the hypersphere surface to $\mathbf{v} = 0$ at infinity.

4. Rotation of a hypersphere

The kinematic basis of rigid-body motion in n-dimensional space is treated in the textbook by Synge & Schild (1949, p. 156). In particular, if a rigid body in hyperspace rotates steadily about a fixed point O, the velocity at any point in the body relative to O is given by $u_j = x_k\,\Omega_{kj}$ where x_k is the position vector of the point relative to O. The angular velocity dyadic $\mathbf{\Omega}$ is a constant skew-symmetric second-rank tensor:

$$\Omega_{jk} = -\Omega_{kj}. \tag{49}$$

[It is only in 3-space that the angular velocity can be represented alternatively as a pseudovector $\boldsymbol{\omega}$, since for $n = 3$ the number of independent components of a skew-symmetric second-rank tensor is three – the same as the number of independent components of a (pseudo)vector (Synge & Schild 1949; Goldstein 1950). This is not the case for any other value of n, except in a special sense for the case $n = 2$.]

Consider a hypersphere rotating about an 'axis' through its centre O, corresponding to $\mathbf{r} = 0$. Since fluid is assumed to adhere to the boundary points of the hypersphere, the boundary condition satisfied at its surface by the fluid velocity vector $\mathbf{v}$ is

$$v_j = x_k\,\Omega_{kj} \quad \text{at} \quad r = a. \tag{50}$$

In addition, boundary conditions (15) and (16) at $r = \infty$ continue to obtain in the present problem.

In consequence of the linearity of the differential equations (12) and (13) and the boundary conditions, we may define (Brenner 1964*a*, *b*, *c*) a triadic 'velocity' field $V_{ijk}(x_1, x_2, \ldots, x_n) \equiv \mathbf{V}(\mathbf{r})$ and a dyadic 'pressure' field $P_{jk}(x_1, x_2, \ldots, x_n) \equiv \mathbf{P}(\mathbf{r})$ via the linear relations

$$v_i = V_{ijk}\,\Omega_{kj} \quad (\mathbf{v} = \mathbf{V}\colon\mathbf{\Omega}), \tag{51}$$

$$p - p_\infty = \mu P_{jk}\,\Omega_{kj} \quad (p - p_\infty = \mu\mathbf{P}\colon\mathbf{\Omega}). \tag{52}$$

The double-dot notation follows the 'nesting' convention of Chapman & Cowling (1970).

Suppose that $T_{\ldots jk}$ is a Cartesian tensor of any rank which is symmetric in its last two indices ('postsymmetric'), i.e. $T_{\ldots jk} = T_{\ldots kj}$. It is easily shown that $T_{\ldots jk}\,\Omega_{kj} = 0$ as a consequence of the skew-symmetric nature of Ω_{kj}. Hence, without loss of generality we may choose

$$V_{ijk} = -V_{ikj}, \quad P_{jk} = -P_{kj}, \tag{53}$$

since only the post-antisymmetric parts of these tensors will contribute to the *physical* quantities v_i and $p - p_\infty$ in (51) and (52).

Substitute (51) and (52) into the differential equations (12) and (13) and the boundary conditions (50) and (15)–(16). Upon eliminating the constant $\mathbf{\Omega}$ from the resulting expressions and using (53), the tensor velocity and pressure fields (V_{ijk}, P_{jk}) are found to satisfy the following differential equations,

$$\nabla^2 V_{ijk} = \partial P_{jk}/\partial x_i, \tag{54}$$

and

$$\partial V_{ijk}/\partial x_i = 0, \tag{55}$$

and boundary conditions,

$$V_{ijk} = \tfrac{1}{2}(\delta_{ij} x_k - \delta_{ik} x_j) \quad \text{at} \quad r = a, \tag{56}$$

$$V_{ijk} \to 0 \quad \text{as} \quad r \to \infty, \tag{57}$$

$$P_{jk} \to 0 \quad \text{as} \quad r \to \infty. \tag{58}$$

As in the comparable translational case it may be concluded from this system of equations and boundary conditions that the fields (V_{ijk}, P_{kj}) are purely geometric in nature, and hence that the geometric symmetries of these fields must be the same as that of the hypersphere itself. As such, the fields $(\mathbf{V}, \mathbf{P})$ must be representable solely in terms of the position vector $\mathbf{r}$ and its tensor invariants.

The only invariant *dyadic* fields which can be derived from $\mathbf{r}$ are the dyadic idemfactor $\mathbf{I}$ and $\mathbf{rr}$, as well as scalar multiples thereof, which can at most be functions of the scalar r. Thus, P_{jk} must possess the same general form as (25). However, in consequence of the skew-symmetric nature of P_{jk} noted in (53), each of these scalar multiples must be identically zero. Accordingly. it follows immediately that

$$P_{jk} = 0. \tag{59}$$

The corresponding triadics which can be formed from $\mathbf{r}$ and its invariants are scalar multiples of†

$$\mathbf{rrr}, \quad \mathbf{Ir}, \quad \mathbf{rI} \quad \text{and} \quad (\mathbf{Ir})^{\dagger} = {}^{\dagger}(\mathbf{rI}),$$

i.e. $x_i x_j x_k$, $\delta_{ij} x_k$, $\delta_{jk} x_i$ and $\delta_{ik} x_j$, respectively. The only combination which can be formed from these four tensors that is antisymmetric in the last pair of indices is the single term $\delta_{ij} x_k - \delta_{ik} x_j$. Hence, it may be concluded on symmetry grounds that $\mathbf{V}$ must be of the form

$$V_{ijk} = (\delta_{ij} x_k - \delta_{ik} x_j) F(r), \tag{60}$$

or, in invariant notation,

$$\mathbf{V} = [\mathbf{Ir} - (\mathbf{Ir})^{\dagger}] F(r), \tag{61}$$

with F a scalar function of $|\mathbf{r}|$ to be determined.

Upon forming the divergence of (60) it is found that the continuity equation (55) is automatically satisfied for any choice of the unknown function $F(r)$.

From (59) and (54) it follows that V_{ijk} satisfies Laplace's equation,

$$\nabla^2 V_{ijk} = 0. \tag{62}$$

With use of (60) this eventually leads to the total differential equation

$$\nabla_r^2 F + \frac{2}{r}\frac{dF}{dr} = 0 \tag{63}$$

† If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are any vectors, the post- and pre-transpose operators are defined by the relations

$$(\mathbf{abc})^{\dagger} = \mathbf{acb} \quad \text{and} \quad {}^{\dagger}(\mathbf{abc}) = \mathbf{bac}.$$

satisfied by F, with ∇_r^2 the radial Laplacian given in (28). This equation is easily solved to obtain the general solution

$$F = C_1 r^{-n} + C_2. \tag{64}$$

The boundary condition at infinity (57) requires that $C_2 = 0$, whereas the boundary condition (56) on the hypersphere surface requires that $C_1 = \frac{1}{2}a^n$. Consequently, the tensor velocity and pressure fields are

$$V_{ijk} = \frac{1}{2}\left(\frac{a}{r}\right)^n (\delta_{ij} x_k - \delta_{ik} x_j) \quad \text{and} \quad P_{jk} = 0, \tag{65}$$

or, in invariant form,

$$\mathbf{V} = \frac{1}{2}\left(\frac{a}{r}\right)^n [\mathbf{I}\mathbf{r} - (\mathbf{I}\mathbf{r})^\dagger] \quad \text{and} \quad \mathbf{P} = 0. \tag{66}$$

From (51) and (52) these lead to the following expressions for the physical velocity and pressure fields:

$$v_i = \left(\frac{a}{r}\right)^n x_j \Omega_{ji}, \quad \text{i.e.} \quad \mathrm{v} = \left(\frac{a}{r}\right)^n \mathbf{r}.\mathbf{\Omega}, \tag{67}$$

and

$$p = p_\infty = \text{const.} \tag{68}$$

4.1. '*Torque*' *on the hypersphere*

The first skew-symmetric moment $\mathbf{T}$ of the stresses acting over the hypersphere surface is defined as

$$T_{jk} = \int_{\Lambda_1} (x_j \, ds_i \, \pi_{ik} - x_k \, ds_i \, \pi_{ij})\Big|_{r=a}. \tag{69}$$

From (45), (67), (68) and (11) a straightforward calculation yields

$$ds_i \, \pi_{il}|_{r=a} = n\mu \Omega_{li}(x_i/r)\, ds \tag{70}$$

upon suppressing the irrelevant constant p_∞ arising from (68). Hence, with use of (7) there follows

$$T_{jk} = n\mu a^n (\delta_{jm}\delta_{kl} - \delta_{jl}\delta_{km})\, \Omega_{li} A_{im}, \tag{71}$$

wherein

$$A_{im} = \int_{\Lambda_1} \frac{x_i x_m}{r^2} d\lambda. \tag{72}$$

This dyadic integral is readily evaluated upon recognizing that as a consequence of the geometric symmetry of the hypersphere it must be isotropic, and hence of the form $A_{im} = \delta_{im} A$. The scalar constant A may be determined by contracting on the indices to obtain $A_{ii} = nA$. Therefore, since $x_i x_i = r^2$, we obtain

$$A = \frac{1}{n}\int_{\Lambda_1} d\lambda = \frac{\Lambda}{n}, \tag{73}$$

with Λ given by (9). Consequently,

$$T_{jk} = -\frac{4\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)} \mu a^n \Omega_{jk}. \tag{74}$$

Alternatively, the antisymmetric stress moment per unit of wetted surface area S is

$$\mathbf{T}/S = -2\mu a \mathbf{\Omega}, \tag{75}$$

upon employing (8) and (9). This result is independent of the dimensionality n of the space. The negative sign is, of course, merely a manifestation of the fact that the 'torque' exerted by the fluid on the hypersphere acts in a 'direction' opposite to that of the rotation.

In the physically important cases of $n = 2$ and $n = 3$ it is possible to express (74) in a more conventional form involving the torque pseudovector $\mathbf{t}$ (Synge & Schild 1949)

$$t_l = \tfrac{1}{2}\epsilon_{ljk} T_{jk}, \tag{76}$$

and the angular velocity pseudovector $\boldsymbol{\omega}$,

$$\omega_l = \tfrac{1}{2}\epsilon_{ljk}\,\Omega_{jk}, \tag{77}$$

with ϵ_{jkl} the usual three-dimensional permutation symbol. Thus, upon multiplying (74) by ϵ_{ljk} and utilizing these definitions, we obtain

$$\mathbf{t} = -\frac{4\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\mu a^n \boldsymbol{\omega} \quad (n = 2 \text{ or } 3). \tag{78}$$

For $n = 2$ and 3, respectively, this yields

$$\mathbf{t} = -4\pi\mu a^2 \boldsymbol{\omega} \quad (n = 2), \tag{79}$$

and

$$\mathbf{t} = -8\pi\mu a^3 \boldsymbol{\omega} \quad (n = 3). \tag{80}$$

These expressions agree with the known results for the circular cylinder (Pai 1956) and sphere (Landau & Lifshitz 1959), respectively, as do the corresponding formulas (67) and (68) for the velocity and pressure fields, since in these cases $\mathbf{r}\,.\,\boldsymbol{\Omega} = \boldsymbol{\omega} \wedge \mathbf{r}$.

5. Faxén's laws for the hypersphere

As in comparable three-dimensional sphere problems (Brenner 1964*c*, 1966*a*, *b*), having solved the problem of uniform translation and rotation it is essentially a trivial matter to obtain expressions for the force and skew-symmetric stress moment for situations involving *arbitrary* boundary conditions $\mathbf{v}$ on the hypersphere surface and arbitrary boundary conditions $\mathbf{v}^\infty$ at infinity. Here, $\mathbf{v}^\infty(\mathbf{r})$ [and $p^\infty(\mathbf{r})$] is itself a solution of the Stokes equations (12) and (13) possessing no singularities within the interior of the space occupied by the hypersphere. An obvious generalization of the analysis of Brenner (1966*a*) yields

$$\mathbf{F} = \frac{n(n-2)\mu}{(n-1)a}\int_{S_n}(\mathbf{v}^\infty - \mathbf{v})\,ds, \tag{81}$$

and

$$\mathbf{T} = \frac{n\mu}{a}\int_{S_n}[\mathbf{r}(\mathbf{v}^\infty - \mathbf{v}) - (\mathbf{v}^\infty - \mathbf{v})\,\mathbf{r}]\,ds, \tag{82}$$

where S_n denotes integration over the surface of the n-dimensional hypersphere of radius a. These formulas express the requisite dynamical entities directly in terms of the prescribed boundary data $\mathbf{v}$ on the hypersphere and the prescribed field $\mathbf{v}^\infty$ at infinity. Their calculation is thus reduced to a quadrature, not requiring the actual solution of the relevant boundary-value problem.

As an elementary application consider the case of the uniform translation of a hypersphere through a fluid at rest at infinity. Thus, we put $\mathbf{v} = \mathbf{U}$ at $r = a$ and

$\mathbf{v}^\infty = 0$. Upon integration (81) yields the value for the force already cited in (48). From (82) the corresponding moment is

$$\mathbf{T} = \frac{n\mu}{a}\left[\mathbf{U}\int_{S_n} \mathbf{r}\,ds - \left(\int_S \mathbf{r}\,ds\right)\mathbf{U}\right].$$

As is easily demonstrated, the symmetry of the hypersphere is such that

$$\int_{S_n} \mathbf{r}\,ds = 0,$$

whence it follows that $\mathbf{T} = 0$. Thus, the translating hypersphere experiences no 'torque' – an expected result. Similarly, by putting $\mathbf{v} = \mathbf{r}\,.\,\mathbf{\Omega}$ [cf. (50)] in (81)–(82), and setting $\mathbf{v}^\infty = 0$, we recover (75) for the 'torque' on a rotating hypersphere. In addition, we arrive at the conclusion that $\mathbf{F} = 0$, showing that a hypersphere rotating about an axis through its centre experiences no force.

In the case where the hypersphere is at rest, equations (81) and (82) reduce to

$$\mathbf{F} = \frac{n(n-2)\mu}{(n-1)a}\int_{S_n} \mathbf{v}^\infty\,ds \tag{83}$$

and

$$\mathbf{T} = \frac{n\mu}{a}\int_{S_n} (\mathbf{r}\mathbf{v}^\infty - \mathbf{v}^\infty\mathbf{r})\,ds. \tag{84}$$

Since $\mathbf{v}^\infty$ possesses no singularities in the interior of the space occupied by the hypersphere it may be expanded in a Taylor series about the hypersphere centre O to obtain

$$\mathbf{v}^\infty = \mathbf{v}^\infty_o + \mathbf{r}\,.\,(\nabla\mathbf{v}^\infty)_o + \frac{1}{2!}\mathbf{r}\mathbf{r}\colon(\nabla\nabla\mathbf{v}^\infty)_o + \dots.$$

In this manner we obtain

$$\int \mathbf{v}^\infty\,ds = \mathbf{v}^\infty_o\int ds + \left(\int \mathbf{r}\,ds\right).(\nabla\mathbf{v}^\infty)_o + \frac{1}{2!}\left(\int \mathbf{r}\mathbf{r}\,ds\right)\colon(\nabla\nabla\mathbf{v}^\infty)_o + \dots.$$

Each of these surface integrals is necessarily isotropic. Since there exist no isotropic tensors of odd rank, we have immediately that

$$\int \mathbf{r}\,ds = 0, \quad \int \mathbf{r}\mathbf{r}\mathbf{r}\,ds = 0, \quad \dots.$$

The isotropic tensors of even rank may be evaluated by the 'contraction' methods employed in connection with equation (72). This yields

$$\int ds = a^{n-1}\Lambda, \quad \int x_i x_j\,ds = \delta_{ij}\,a^{n+1}\frac{\Lambda}{n},$$

$$\int x_i x_j x_k x_l\,ds = (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\frac{\Lambda}{n(n+2)}, \quad \dots.$$

Consequently (see also the appendix),

$$\int_{S_n} \mathbf{v}^\infty\,ds = a^{n-1}\Lambda\left[\mathbf{v}^\infty_o + \frac{a^2}{2!n}(\nabla^2\mathbf{v}^\infty)_o + \frac{1.3a^4}{4!n(n+2)}(\nabla^4\mathbf{v}^\infty)_o + \frac{1.3.5a^6}{6!n(n+2)(n+4)}(\nabla^6\mathbf{v}^\infty)_o + \dots\right]. \tag{85}$$

Now, $\mathbf{v}^\infty$ satisfies the Stokes equations,

$$\nabla^2 \mathbf{v}^\infty = (1/\mu)\nabla p^\infty, \quad \nabla . \mathbf{v}^\infty = 0. \tag{86}$$

These show that

$$\nabla^2 p^\infty = 0,$$

and hence that

$$\nabla^{2m}\mathbf{v}^\infty = 0 \quad (m \geqslant 2).$$

Thus, the infinite series in brackets in (85) terminates after only two terms. In this manner (83) adopts the form

$$\mathbf{F} = \frac{n(n-2)\,2\pi^{\frac{1}{2}n}}{(n-1)\,\Gamma(\frac{1}{2}n)}\mu a^{n-2}\left[\mathbf{v}_o^\infty + \frac{a^2}{2n\mu}(\nabla p^\infty)_o\right]. \tag{87}$$

A similar analysis of (84) eventually yields the single term

$$\mathbf{T} = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\mu a^n[\nabla\mathbf{v}^\infty - (\nabla\mathbf{v}^\infty)^\dagger]_o. \tag{88}$$

Termination of the infinite series that would otherwise have appeared in (88) is based upon the fact that

$$\nabla^{2m}[\nabla\mathbf{v}^\infty - (\nabla\mathbf{v}^\infty)^\dagger] = 0 \quad (m \geqslant 1). \tag{89}$$

This relation follows by taking the gradient of the first of equations (86) to obtain

$$\nabla^2\nabla\mathbf{v}^\infty = (1/\mu)\,\nabla\nabla p^\infty.$$

Upon forming the transpose of this identity and subtracting from the above we thereby obtain (89) for the case $m = 1$. The proof for $m > 1$ then follows by successive applications of the operator ∇^2 to the case $m = 1$.

For a space of $n = 3$ dimensions, the generalized Faxén laws (87) and (88) reduce to their usual forms (Brenner 1964*b*, 1966*a*, *b*),

$$\mathbf{F} = 6\pi\mu a\left[\mathbf{v}_o^\infty + \frac{a^2}{6\mu}(\nabla p^\infty)_o\right],$$

and

$$\mathbf{t} = 8\pi\mu a^3 \boldsymbol{\omega}_o^\infty,$$

where $\boldsymbol{\omega}^\infty = \frac{1}{2}\nabla \wedge \mathbf{v}^\infty$.

6. Stream function for a translating hypersphere

The velocity field (38) for a translating hypersphere can be expressed alternatively in terms of an n-dimensional stream function. We deliberately refrained from introducing this concept in § 3, where it should otherwise naturally have appeared, in order to emphasize clearly the geometric nature of the symmetry arguments advanced there. Thus, we were able clearly to demonstrate the co-ordinate-free nature of our techniques by not confusing use of the position vector $\mathbf{r}$ or scalar $|\mathbf{r}|$ with comparable variables appearing in the generalization of spherical co-ordinates to n dimensions. Nor should our use of Cartesian tensors suggest either that we have utilized a *Cartesian* co-ordinate system in resolving the fundamental boundary-value problems posed. Their introduction was purely pragmatic in the sense that the entire geometric analysis could have been performed entirely in invariant $\mathbf{r}$ form, which is consistent with the spirit of the present contribution.

Despite this purist stance,† it would be overstating the case to claim that no advantages exist to utilizing a convenient co-ordinate parameterization that readily lends itself to the geometry of the hypersphere. Towards this end introduce a system of n-dimensional hyperspherical polar co-ordinates (Bateman 1944; Sommerfeld 1964) $(r, \theta, \phi_1, \phi_2, \ldots, \phi_{n-2})$ defined relative to a Cartesian system via the relations

$$\left.\begin{aligned} x_1 = r\cos\theta, \quad x_2 = r\sin\theta\cos\phi_1, \quad x_3 = r\sin\theta\sin\phi_1\cos\phi_2, \quad \ldots,\\ x_{n-1} = r\sin\theta\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-3}\cos\phi_{n-2},\\ x_n = r\sin\theta\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-3}\sin\phi_{n-2}. \end{aligned}\right\} \tag{90}$$

In order to span all points in space, $-\infty < x_k < \infty\ (k = 1, 2, \ldots, n)$, the appropriate range of these co-ordinates is $0 < r < \infty$, $0 < \theta < \pi$, $0 < \phi_j < \pi\ (j = 1, 2, \ldots, n-3)$ and $0 < \phi_{n-2} < 2\pi$.

With
$$q_1 = r, \quad q_2 = \theta, \quad q_3 = \phi_1, \quad \ldots, \quad q_{n-1} = \phi_{n-3}, \quad q_n = \phi_{n-2}, \tag{91}$$
a system of curvilinear co-ordinates (Happel & Brenner 1973), the metrical coefficients

$$h_k = |dq_k/dl_k| \quad \text{(no sum on } k\text{)}$$

(with dl_k a line element along the q_k-co-ordinate curve) may be computed from the formulae

$$\frac{1}{h_k^2} = \sum_{j=1}^{n}\left(\frac{\partial x_j}{\partial q_k}\right)^2 \quad (k = 1, 2, \ldots, n). \tag{92}$$

This yields

$$\begin{aligned} h_1 = 1, \quad h_2 = 1/r, \quad h_3 = 1/r\sin\theta, \quad h_4 = 1/r\sin\theta\sin\phi_1, \quad \ldots,\\ h_n = 1/r\sin\theta\sin\phi_1\ldots\sin\phi_{n-3} \end{aligned} \tag{93}$$

for the metrical coefficients.

The vector (no sum on k)

$$\hat{\mathbf{q}}_k = h_k\frac{\partial \mathbf{r}}{\partial q_k} \equiv h_k \sum_{j=1}^{n} \hat{\mathbf{x}}_j \frac{\partial x_j}{\partial q_k} \quad (k = 1, 2, \ldots, n) \tag{94}$$

represents a unit tangent vector to the q_k-co-ordinate curve in the direction of algebraically increasing values of q_k. The hyperspherical polar system may be demonstrated to be an orthogonal system in the sense that

$$\hat{\mathbf{q}}_k \cdot \hat{\mathbf{q}}_l = \delta_{kl}, \tag{95}$$

with δ_{kl} the Kronecker delta – a fact that may be confirmed by observing from the definitions (90) and (91) that

$$\frac{\partial \mathbf{r}}{\partial q_k}\cdot\frac{\partial \mathbf{r}}{\partial q_l} = \sum_{j=1}^{n}\left(\frac{\partial x_j}{\partial q_k}\right)\left(\frac{\partial x_j}{\partial q_l}\right) = 0$$

for each indicial pair $(k, l) = 1, 2, 3, \ldots, n\ (k \neq l)$.

† To satisfy wholly the purist, it is necessary to demonstrate that (28) can be derived in a co-ordinate-free manner. This is readily done by noting that, with

$$r = |\mathbf{r}|, \quad \nabla f(r) = (\partial/\partial\mathbf{r})f(r) = \mathbf{r}r^{-1}\,df/dr,$$

where f is any function depending only upon r. Hence, in succession,

$$\begin{aligned} \nabla^2 f &= \nabla.\nabla f = \nabla.(\mathbf{r}r^{-1}\,df/dr)\\ &= (\nabla.\mathbf{r})\,r^{-1}\,df/dr + \mathbf{r}.\nabla(r^{-1}\,df/dr). \end{aligned}$$

Since $\nabla.\mathbf{r} = n$ and $\mathbf{r}.\nabla \equiv r\partial/\partial r$, this yields

$$\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{n-1}{r}\frac{df}{dr},$$

which is equivalent to (28).

Inverse to (94) for such an orthogonal system is the generic relation (Happel & Brenner 1973)

$$\hat{\mathbf{x}}_k = \sum_{j=1}^{n} \hat{\mathbf{q}}_j h_j \frac{\partial x_k}{\partial q_j} \quad (k = 1, 2, \ldots, n), \tag{96}$$

which permits each of the Cartesian unit vectors $\hat{\mathbf{x}}_k$ to be explicitly expressed in terms of the curvilinear unit vectors $\hat{\mathbf{q}}_j$. Of particular interest is the case $k = 1$, which yields

$$\hat{\mathbf{x}}_1 = \hat{\mathbf{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta. \tag{97}$$

For convenience choose the translational velocity vector $\mathbf{U}$ of the hypersphere in (14) to lie along the polar axis x_1. Thus, $\mathbf{U} = \hat{\mathbf{x}}_1 U$ where U is given by (40). Consequently, in (41) $\hat{\mathbf{U}} \equiv \hat{\mathbf{x}}_1$, whence from (97)

$$\hat{\mathbf{U}} = \hat{\mathbf{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta. \tag{98}$$

Hence, the angle θ may be defined in invariant form as

$$\theta = \cos^{-1}(\hat{\mathbf{r}} \cdot \hat{\mathbf{U}}) \quad (0 \leqslant \theta < \pi), \tag{99}$$

since from (95) $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$. Introduce (98) into (38) to obtain

$$\mathbf{v} = \hat{\mathbf{r}} v_r + \hat{\boldsymbol{\theta}} v_\theta, \tag{100}$$

where

$$v_r = \frac{U \cos\theta}{2} \left(\frac{a}{r}\right)^n \left[n \left(\frac{r}{a}\right)^2 - (n-2) \right], \tag{101}$$

$$v_\theta = -\frac{U \sin\theta}{2(n-1)} \left(\frac{a}{r}\right)^n \left[n \left(\frac{r}{a}\right)^2 + n - 2 \right]. \tag{102}$$

Observe that the remaining $(n-2)$ velocity components, $v_{\phi_k} = \hat{\boldsymbol{\phi}}_k \cdot \mathbf{v}$ $(k = 1, 2, \ldots, n-2)$ are such that

$$v_{\phi_1} = v_{\phi_2} = \ldots = v_{\phi_{n-2}} = 0 \tag{103}$$

in present circumstances.

In view of the latter condition in conjunction with the fact that the non-zero velocity components v_r and v_θ are functions only of (r, θ), and hence independent of $\phi_1, \phi_2, \ldots, \phi_{n-2}$, it is possible to represent the velocity field via an n-dimensional stream function, $\psi(r, \theta)$. That this is indeed the case may be seen as follows. The generic expression for the divergence of a vector field in orthogonal curvilinear co-ordinates is (Happel & Brenner 1973)

$$\nabla \cdot \mathbf{v} = H \sum_{j=1}^{n} \frac{\partial}{\partial q_j} \left(\frac{h_j v_j}{H} \right),$$

with $v_j = \hat{\mathbf{q}}_j \cdot \mathbf{v}$ and $H = h_1 h_2 \ldots h_n$. Consequently, in hyperspherical polar co-ordinates, with $\mathbf{v}$ given by (100) [and conditions (103) prevailing], we have in present circumstances that

$$\nabla \cdot \mathbf{v} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} (r^{n-1} v_r) + \frac{1}{r \sin^{n-2}\theta} \frac{\partial}{\partial \theta} (\sin^{n-2}\theta \, v_\theta).$$

The continuity equation $\nabla \cdot \mathbf{v} = 0$ will therefore be satisfied automatically by any choice of a function $\psi(r, \theta)$ such that

$$v_r = -\frac{1}{r^{n-1} \sin^{n-2}\theta} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r^{n-2} \sin^{n-2}\theta} \frac{\partial \psi}{\partial r}. \tag{104}$$

With v_r and v_θ given explicitly by (101) and (102), it follows upon integration that the stream function is

$$\psi = \frac{Ua^{n-1}}{2(n-1)}\sin^{n-1}\theta\left[(n-2)\left(\frac{a}{r}\right)-n\left(\frac{r}{a}\right)\right], \tag{105}$$

to within an arbitrary additive constant. For completeness we note that in this notation the pressure field $p(r,\theta)$ obtained from (39) is

$$p = p_\infty + \frac{n(n-2)}{(n-1)}\frac{\mu U\cos\theta}{a}\left(\frac{a}{r}\right)^{n-1}. \tag{106}$$

For case $n = 3$ the above relation reduces to the well-known Stokes stream function for flow around a sphere (Happel & Brenner 1973).

The surfaces $\psi(r,\theta) = \text{constant}$ define the stream surfaces. Thus, the streamlines of the flow lie simultaneously in the 'meridian planes' $\phi_1 = \text{constant}, \phi_2 = \text{constant}, \ldots, \phi_{n-2} = \text{constant}$. This fact is consistent with equations (103), which show that no fluid motion occurs across any 'meridian plane'. It is interesting to note that the powers of r appearing in the expression (105) for the stream function are explicitly independent of the dimensionality n of the space.

The stream function for a uniform streaming flow with velocity U in the negative x_1 direction is

$$\psi = \frac{U}{n-1}(r\sin\theta)^{n-1}. \tag{107}$$

Addition of this to (105) yields the stream function† for streaming flow in the negative x_1 direction with velocity U past a *stationary* hypersphere. The corresponding pressure field continues to be given by (106).

7. Discussion

The addition of a uniform stream $-\mathbf{U}$ to (38) gives the velocity field, $\mathbf{v}'$, say, for streaming flow with velocity $-\mathbf{U}$ past a stationary hypersphere. At large distances, $r/a \to \infty$, the velocity field appropriate to this problem is asymptotically of the form

$$\mathbf{v}' \sim -U + UO(a/r)^{n-2}.$$

From this may be computed the ratio of inertial to viscous forces:

$$\frac{\rho|\mathbf{v}'.\nabla\mathbf{v}'|}{\mu|\nabla^2 v'|} \sim \frac{R(r/a)}{n-1}$$

for $n \neq 2$. Here, $R = Ua/\nu$. Thus, just as in the $n = 3$ case (Proudman & Pearson 1957), Stokes' solution is not a uniformly valid solution of the full Navier–Stokes equations in the limit where $R \to 0$. Rather, Stokes' solution becomes invalid at distances r/a exceeding

$$\frac{r}{a} = O\left(\frac{n-1}{R}\right).$$

† One of the referees kindly brought to my attention the fact that this n-dimensional stream function has already been given by Burns (1969) as an application of GASPT (generalized axially symmetric potential theory). In this context it is worth emphasizing that our general geometrical methods for solving hypersphere boundary value problems are not limited to such axially symmetric flows. For example, the geometrical scheme may be used to solve the non-axisymmetric problem of shearing flow past a hypersphere (cf. Brenner 1964*b* for the comparable $n = 3$ case) by introducing into (51) and (52) the symmetric rate-of-strain tensor $S_{kj} = S_{jk}$ in place of the antisymmetric angular velocity tensor $\Omega_{kj} = -\Omega_{jk}$.

Whitehead's paradox therefore always obtains irrespective of the dimensionality n. However, since n figures explicitly in the above criterion, for a fixed Reynolds number the nature of the singularity is the weaker the larger is the dimensionality n of the space.

Our emphasis in this paper has been concerned exclusively with viscous flow at asymptotically small Reynolds numbers. However, n-dimensional fluid mechanics provides some interesting possibilities for the complete Navier–Stokes equations, valid at all Reynolds numbers, albeit in spaces of dimensionality $n > 3$.

All known non-trivial solutions of the complete Navier–Stokes equations derive, in essence, from the existence of special similarity transformations. In general such transformations obtain only in a space of one particular dimensionality n, but no others. Thus, for example, there exists an exact solution (cf. Landau & Lifshitz 1959, pp. 86–88) of the complete Navier–Stokes equations for an isolated point-force singularity in a fluid at rest at infinity for the case of $n = 3$, but not for $n = 2$. Conversely, for $n = 2$ there exists an exact solution for a point couple singularity,‡ but not for $n = 3$. As a final example, we note that for $n = 2$ there exists an exact solution of the Navier–Stokes equations (cf. Landau & Lifshitz 1959, pp. 81–86) for radial flow in which the stream function $\psi(\theta)$ is independent of r – corresponding to Jeffery–Hamel flow from or towards a point source at the apex of two intersecting planes. No comparable exact solution is known, however, for $n = 3$ – corresponding to the circular cone (Goldstein 1965; Ackerberg 1965). Clearly, by expanding the domain of the search beyond three dimensions the possibility exists of deriving many more solutions of the full Navier–Stokes equations via such similarity arguments. In turn, this allows the use of perturbation methods involving a small parameter ϵ to extend these new exact solutions from n backwards to $n - |\epsilon|$ dimensions – that is, backwards towards the usual domains $n = 3$ and $n = 2$ of physical interest. Such perturbation methods have found wide use in other fields of physics, as outlined in the Introduction, and there exists every reason to suppose that such methods will also find extensive applications to viscous flow problems.

A companion paper (Brenner 1981) presents the solution for Poiseuille flow through an n-dimensional hypercylinder.

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Appendix. Symbolic operators in n-dimensional space

Equation (85) may also be derived via symbolic-operator methods (Brenner 1966*a*, *b*) upon employing the symbolic Taylor-series expansion

$$\mathbf{v}^{\infty} = \exp(\mathbf{r}\,.\,\boldsymbol{\nabla}_o)\,\mathbf{v}_o^{\infty}.$$

Define the (symbolic) angle θ as $\mathbf{r}\,.\,\boldsymbol{\nabla}_o = (r\nabla_o)\cos\theta$, where $\nabla_o = (\nabla_o^2)^{\frac{1}{2}}$ with ∇_o^2 the Laplace operator in $\mathbf{r}_o$-space. In the notation of § 6, the scalar element of surface area on a hypersphere of radius r is given by (7), wherein

$$d\lambda = \sin^p\theta\, d\Phi,$$

‡ This simple solution represents the limiting case of the steady rotation of a circular cylinder (Pai 1956), from which it is easily derived by allowing the cylinder radius to tend to zero while allowing the couple to remain non-zero.

in which $p = n-2$ and

$$d\Phi = \sin^{p-1}\phi_1 \sin^{p-2}\phi_2 \ldots \sin^2\phi_{p-2}\sin\phi_{p-1}\, d\phi_1\, d\phi_2 \ldots d\phi_{p-1}\, d\phi_p.$$

Therefore,

$$\int_{r=a} ds \exp(\mathbf{r}\,.\,\boldsymbol{\nabla}_o) = a^{p+1}\left(\int_0^\pi \exp(a\nabla_o \cos\theta)\sin^p\theta\, d\theta\right)\Phi,$$

in which (Sommerfeld 1964)

$$\Phi = \int d\Phi = \frac{2\pi^{(p+1)/2}}{\Gamma\left(\dfrac{p+1}{2}\right)}.$$

However (Gradshteyn & Ryzhik 1965), upon putting $2v = p = n-2$,

$$\int_0^\pi \exp(\pm\beta\cos\theta)\sin^{2v}\theta\, d\theta = \pi^{\frac{1}{2}}\left(\frac{2}{\beta}\right)^v \Gamma(v+\tfrac{1}{2})\, I_v(\beta)$$

($\mathrm{Re}\, v > -\frac{1}{2}$), for any constant parameter β, wherein I_v is the modified Bessel function of the first kind. In this manner one obtains

$$\int_{r=a} ds \exp(\mathbf{r}\,.\,\boldsymbol{\nabla}_o) = 2\pi^{v+1}a^{2v+1}\left(\frac{2}{a\nabla_o}\right)^v I_v(a\nabla_o),$$

with $v = \frac{1}{2}n - 1$. Upon utilizing the infinite series representation of the modified Bessel function (Abramowitz & Stegun 1964) this becomes

$$\int_{r=a} ds \exp(\mathbf{r}\,.\,\boldsymbol{\nabla}_o) = 2\pi^{\frac{1}{2}n}a^{n-1}\sum_{k=0}^{\infty}\frac{(a^2\nabla_o^2/4)^k}{k!\,\Gamma(k+\frac{1}{2}n)}.$$

When applied to the vector field $\mathbf{v}_o^\infty$ this result is equivalent to (85).

The comparable integral appearing in the stress moment may be derived in a similar manner by utilizing the identity

$$\int_{r=a} ds\, \mathbf{r} \exp(\mathbf{r}\,.\,\boldsymbol{\nabla}_o) \equiv \frac{\partial}{\partial\boldsymbol{\nabla}_o}\int_{r=a} ds \exp(\mathbf{r}\,.\,\boldsymbol{\nabla}_o),$$

in conjunction with

$$(\partial/\partial\boldsymbol{\nabla}_o)(\nabla_o^2)^k = 2k(\nabla_o^2)^{k-1}\boldsymbol{\nabla}_o.$$

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